20080829 Quiver varieties and double affine Grassmannian

○ Review of geometric Satake correspondence

G : reductive grp $/ \mathbb{C}$

$\mathcal{K} = \mathbb{C}((s)) \supset \mathcal{O} = \mathbb{C}[[s]]$

$Gr_G = G(\mathcal{K}) / G(\mathcal{O})$: affine Grassman

$G(\mathcal{O})$-orbits on Gr_G

$\longleftrightarrow \lambda \in \Lambda^+ =$ coweight lattice of $G = \mathrm{Hom}(\mathbb{G}_m, T)$

$=$ weight lattice of ${}^L G$

$$Gr_G = \coprod_{\lambda \in \Lambda^+} Gr_G^\lambda \quad : \text{stratification}$$

$IC(\overline{Gr}^\lambda)$: IC sheaf of $\overline{Gr_G^\lambda}$

$\mathcal{P} = \mathrm{Perv}_{G(\mathcal{O})} Gr_G$: abelian category of $G(\mathcal{O})$-equiv perv. sheaves on Gr

It has a tensor structure via convolution diagram

$$G(\mathcal{K}) \times_{G(\mathcal{O})} Gr_G = Gr_G \tilde{\times} Gr_G \xrightarrow{\varpi} Gr_G$$

Gr_G-b'dle over Gr_G

$A * B := \varpi_* (A \tilde{\boxtimes} B)$

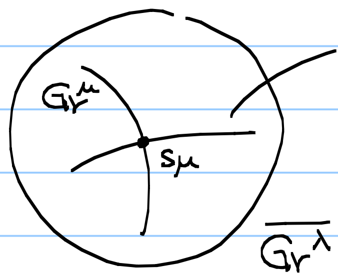
Th. $(\mathcal{P}, *) \cong (\mathrm{Rep}(G^\vee), \otimes)$ as $\otimes$-categories $\left(\begin{array}{c} s^\mu \in \overline{Gr^\lambda} \\ \Leftrightarrow \ \mu \leq \lambda \end{array} \right)$

$IC(\overline{Gr^\lambda}) \leftrightarrow V(\lambda)$: h.w $= \lambda$

and

stalk of $IC(\overline{Gr^\lambda})$ at $s^\mu \leftrightarrow V(\lambda)_\mu$: weight space

- more suitable for double affine generalization



$\overline{M}^\lambda_\mu$: transversal slice to Gr^μ in $\overline{Gr^\lambda}$

$\overline{M}^\lambda_\mu \cap Gr^\lambda =: M^\lambda_\mu \underset{\text{open}}{\subset} \overline{M}^\lambda_\mu$

$V(\lambda)_\mu \cong IC(\overline{Gr^\lambda})$ at $s^\mu \cong IC(\overline{M}^\lambda_\mu)$ at 0

Question

What is the affine analog of the affine Grassmann
$=$ double affine Grassmann ?

$V(\lambda)$: ∞-dimensional
$V(\lambda) \otimes V(\mu)$: ∞-direct sum of $V(\nu)$'s

Consider only integrable highest weight rep.

<u>Proposal</u> (Braverman - Finkelberg 0711.2083)

analog of $\overline{M}^\lambda_\mu$ = Uhlenbeck partial cptification of G-instantons on $\mathbb{R}^4/\mathbb{Z}_l$

l = level of the rep. of aff. KM group

$H^*(IC(\text{analog of } \overline{M}^\lambda_\mu)) \cong V(\lambda)_\mu$ rep. of $(G_{\text{aff}})^\vee$

certain diagram explained later ⟷ $\otimes$

G : simple & simply-connected

$\mathrm{Bun}_G^k(\mathbb{C}^2)$ = framed moduli space of G_{cpt}-instantons on S^4 with $c_2 = k$

trivialization at ∞

= framed moduli space of algebraic G-b'dles on $\mathbb{CP}^2$

trivialization at $l_\infty \subset \mathbb{CP}^2$

smooth of dim $= 2kh^\vee$

$$\mathrm{Bun}_G^k(\mathbb{C}^2) \subset \mathcal{U}_G^k(\mathbb{C}^2) := \coprod_{0 \leq k' \leq k} \mathrm{Bun}_G^{k'}(\mathbb{C}^2) \times S^k \mathbb{C}^2$$

Uhlenbeck cptification

Fix a hom $\mu : \mathbb{Z}_l \to G$

$\cap$

$SL(2) \subset GL(2)$

$$\mathbb{Z}_l \curvearrowright \mathrm{Bun}_G^k(\mathbb{C}^2) \subset \mathcal{U}_G^k(\mathbb{C}^2)$$

through the action of diagonal emb. to $(\mathrm{incl} \times \mu) : \mathbb{Z}_l \to GL(2) \times G$

fixed pts $=: \mathrm{Bun}_G^k(\mathbb{C}^2/\mathbb{Z}_l)$

another inv. $\lambda : \mathbb{Z}_\ell \longrightarrow G$ from. action corr. to the fiber at $0 \in \mathbb{C}^2$

$\mathcal{U}^{\lambda,d}_{G,\mu}$:= fixed pt set

Technical conjecture

$\mathcal{U}^{\lambda,d}_{G,\mu}$: irreducible

Lemma.

$\lambda, \mu \in \mathrm{Hom}(\mathbb{Z}_\ell, G)/_{\text{conj.}} \overset{\text{bijection}}{\longleftrightarrow}$ level ℓ wts of $(\hat{\mathfrak{g}})^\vee$

$\hat{\mathfrak{g}}^\vee$ does not contain the degree operator d

lifts to $(\mathfrak{g}_{\mathrm{aff}})^\vee$: unique up to $\mathbb{C}\delta$

$\tilde{\lambda}, \tilde{\mu}$: lifts s.t. $\langle \tilde{\lambda} - \tilde{\mu}, d \rangle = \underset{\overset{\|}{c_2}}{k}\,\ell$

Main Conjecture 1

$$H^*(IC(\mathcal{U}^{\lambda,d}_{G,\mu})) = V(\tilde{\lambda})_{\tilde{\mu}}$$

Rem

① $\mathcal{V}(\tilde{\lambda}+c\delta)_{\tilde{\mu}+c\delta} \cong \mathcal{V}(\tilde{\lambda})_{\tilde{\mu}}$

② $\exists$ graded version

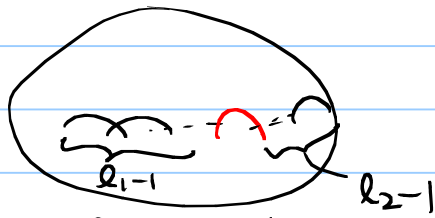
LHS : cohomological grading

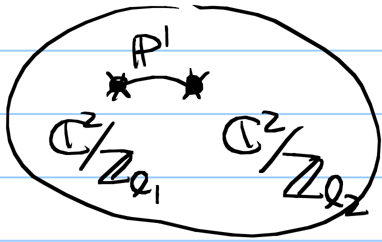
RHS : principal nilpotent

Tensor product

$\ell = \ell_1 + \ell_2$

partial resolution

$\mathbb{C}^2/\mathbb{Z}_\ell \longleftarrow X_{\ell_1,\ell_2} \longleftarrow$ 



$(\ell-1)$ $\mathbb{P}^1$'s
$= (\ell_1-1)+(\ell_2-1)+1$

Consider Uhlenbeck space on X_{ℓ_1,ℓ_2}

$\mathcal{U}^{\lambda_1,\lambda_2,d}_{G,\mu}$ $\qquad \lambda_1, \lambda_2 : \mathbb{Z}/\ell_1, \mathbb{Z}/\ell_2 \longrightarrow G$

level ℓ_1, ℓ_2 weights

Technical conjecture

$\exists$ semismall morphism $\pi : \mathcal{U}_{G,\mu}^{\lambda_1,\lambda_2,d}(X_{\ell_1,\ell_2}) \to \mathcal{U}_{G,\mu}^{\lambda,d}(\mathbb{C}^2)$, $\lambda = \lambda_1+\lambda_2$

Main Conjecture 2

$$\pi_* IC(\mathcal{U}_{G,\mu}^{\lambda_1,\lambda_2,d}(X_{\ell_1,\ell_2})) = \bigoplus IC(\mathcal{U}_{G,\mu}^{\lambda',d})^{\oplus m_{\lambda_1\lambda_2}^{\lambda'}} \oplus \text{other}$$

with $$(V(\lambda_1)\otimes V(\lambda_2))_\mu = \bigoplus_{\lambda'} V(\lambda')_\mu^{\oplus m_{\lambda_1,\lambda_2}^{\lambda'}}$$

Th. conjectures (except graded version of MC1) are true for $G = SL(r)$

$G = SL(r)$ ---- $\mathcal{U}_{G,\mu}^{\lambda,d}$ is an (affine) quiver variety

its IC sheaf was computed

---- related to rep. theory of $\widehat{\mathfrak{sl}}_\ell$ at level $= r$

I. Frenkel level-rk duality

$$\widehat{sl}(r)_\ell \longleftrightarrow \widehat{sl}(\ell)_r$$

$\otimes$ branching to $\widehat{sl}(\ell_1)_r \oplus \widehat{sl}(\ell_2)_r$

I develop the theory for the branching in the quiver variety

<u>Rem</u> ① technical advantage for $G = SL(r)$
$\exists$ nice resolution of $\mathcal{U}_{\hat{G},\mu}$

(Gieseker cptification)

② quiver variety generalization to other $\Gamma \subset SL(2)$
$\leftrightarrow$ affine ADE

<u>Question</u>. G_{E_8} - instantons on Γ_{E_8} ?